GK-E&B

A Gyrokinetic Simulation Model for Low-Frequency Electromagnetic Fluctuations in Magnetized Plasmas †

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Outlines

(I) Introduction

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(I) Introduction

⊙ Low-frequency ($|\omega| \ll |\Omega_{ci}|$) e.m. fluctuations are prevalent in nature and laboratory magnetized plasmas:

⇒ Alfvén waves in solar plasmas, corona heating, solar wind turbulence

⇒ Alfvén waves and instabilities in fusion plasmas

⊙ Disparate [$\sim O(10^3)$] spatial and temporal scales ⇒ $|\rho_i/a| \lesssim O(10^{-3})$

⇒ $\omega_{TAE} \sim O(10^2)\gamma_L$, $\gamma_L \sim \gamma_{NL} \sim O(10)\,\gamma_{tearing}$

⊙ Microscopic (ρ_i)-scale physics + wave-particle resonances/interactions

⇒ $\boxed{\text{nonlinear}}$ $\boxed{\text{(1st-principle based)}}$ gyrokinetic simulation necessary for accurately assess nonlinear dynamics and transports.

◎ Conventional GK simulations use potentials ($\delta\phi$, $\delta\underset{\sim}{A}$) ⇒ Intensive computations in realistic plasma configurations

⇒ Our model : GK-E&B

- Use $\underset{\sim}{E}$ and $\underset{\sim}{B}$ directly for GK equations
- Accurate descriptions of finite-ρ_i effects and linear and nonlinear wave-particle interactions

⇒ Applications to a broad scope of magnetized plasmas

R3B

⊙ History

- Gyrokinetic electron & Fully kinetic ion Model
  - ⇒ GeFi: [2005 ⋮ 2008...] : $|\omega| \lesssim \omega_{eh}, \omega_{whistler} \ll |\Omega_e|$

    $|k_\perp \rho_e|, |k_\perp \lambda_{De}| \lesssim 1$
  - ⇒ Ge : $\delta\phi, \delta\underset{\sim}{A}$
  - ⇒ Fi (Vlasov ion) : $\delta\underset{\sim}{E}, \delta\underset{\sim}{B}$
  - ⟹ $(\delta\phi, \delta A_\parallel, \delta B_\parallel)$ field variables
  - ⟹ field equations : Set of "coupled" Poisson's eqns
  - ⟹ ?? what about Ge with $\delta\underset{\sim}{E}$ & $\delta\underset{\sim}{B}$ as field variables??

⇒ ○ NLGKE – E & B : PPCF(2019) and PST(2020)

⇒ $\boxed{\text{GeFi} - E \,\&\, B}$

⇒ $\boxed{\text{GK} - E\&B \;\; [2020-2021]}$

(II) Theoretical Formulation

(A) Vlasov equation (collisionless for now)

$$\boxed{\left[\partial_t + \nabla\cdot\underset{\sim}{v} + \frac{q}{m}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B}/c)\cdot\partial_{\underset{\sim}{v}}\right] f(\underset{\sim}{x},\underset{\sim}{v},t) = 0}, \quad (1)$$

$\langle m v f\rangle_v \Rightarrow n m \underset{\sim}{u}$ $\quad$ $\underset{\sim}{P} = \langle m \underset{\sim}{v}\underset{\sim}{v} f\rangle_v$

$$\Rightarrow \boxed{\partial_t (nm\underset{\sim}{u})_j + \nabla\cdot\underset{\sim}{P}_j - n_j q_j(\underset{\sim}{E} + \underset{\sim}{u}_j\times\underset{\sim}{B}/c) = 0} \quad (2)$$

◎ Assumptions: (i) quasi-neutrality condition

$$\left.\begin{array}{l} \boxed{\sum_j n_j q_j \cong 0} \Leftarrow |k\lambda_D|^2 \ll 1 \\ \text{(ii)}\ \boxed{m_i >> m_e} \end{array}\right\} \quad (3)$$

◎ Summing up all species; e + ions

[Single ion for the talk, multiple ion species in the manuscript] $\Rightarrow$ total $\perp \underset{\sim}{B}$ momentum conservation

$$\Rightarrow \boxed{\partial_t (\rho_m \underset{\sim}{u}_{i\perp}) = \frac{1}{c}\underset{\sim}{J}\times\underset{\sim}{B} - [\nabla\cdot\underset{\sim}{P}]_\perp} \quad (4)$$

∘ ρ_m : ion mass only $\quad$ ∘ $\underset{\sim}{P} = \sum_{j=e,i} \underset{\sim}{P}_j$

∘ m_e ignored: valid only in the $\perp \underset{\sim}{B}$ direction

[Since $|\underset{\sim}{u}_{e\perp}|/|\underset{\sim}{u}_{i\perp}| \sim O(1)$]

- m_e : important in $\parallel \underset{\sim}{B}$ dynamics $\Rightarrow$ e.g., electron Landau damping

◎ Given $\underset{\sim}{B}(\underset{\sim}{x}, t) \Rightarrow \boxed{\underset{\sim}{J} = \frac{c}{4\pi} \underset{\sim}{\nabla} \times \underset{\sim}{B}} \Leftarrow (|\omega| \ll |ck|)$ (5)

◎ Given $f_j(\underset{\sim}{x}, \underset{\sim}{v}, t) \Rightarrow \underset{\sim}{P}_j, n_j, u_{j\parallel}$ by NLGKE (to be described later)

(4) $\Rightarrow$ advance $\underset{\sim}{u}_{i\perp}$

◎ In order to advance $f_j \Rightarrow$ need to determine $\underset{\sim}{E}(\underset{\sim}{x}, t)$

- Ion $\perp$ momentum balance $|\omega| \ll |\Omega_{ci}|$

$$\Rightarrow \boxed{\underset{\sim}{E}_\perp = -\frac{1}{c} \underset{\sim}{u}_{i\perp} \times \underset{\sim}{B} + \frac{1}{n_i q_i} [\underset{\sim}{\nabla} \cdot \underset{\sim}{P}_i]_\perp} \quad (6)$$

[Gyrokinetic ion $\perp$ Ohm's Law]

[massless electron model : not valid for determining $E_\parallel$]

How to determine $E_\parallel$? [PPCF 2019]

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◎ $\boxed{\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}}$ $\quad |\omega| \ll |ck|$

$\mathbf{b}\cdot\frac{\partial}{\partial t} \Rightarrow \mathbf{b}\cdot\nabla\times\partial_t\mathbf{B} = \mathbf{b}\cdot\nabla\times(-c\nabla\times\mathbf{E})$

$$\Rightarrow \boxed{c^2[\nabla_\perp^2 E_\parallel - \mathbf{b}\cdot\nabla(\nabla\cdot\mathbf{E}_\perp)] = 4\pi\partial_t J_\parallel[E_\parallel]} \quad (7)$$

◎ $\partial_t J_{\parallel j}$ given by $f_j \Rightarrow E_\parallel$ via parallel (to $\mathbf{B}$) momentum conservation

◎ $\boxed{\mathbf{E} = \mathbf{E}_\perp + E_\parallel \mathbf{b}}$ determined

◎ $\boxed{\partial_t \mathbf{B} = -c\nabla\times\mathbf{E}} \Rightarrow$ advance $\mathbf{B}$ $\quad (8)$

◎ $\mathbf{E}$ and $\mathbf{B} \Rightarrow$ advance $f_j(\mathbf{x}, \mathbf{v}, t)$

◎ So far exact except

$$\left.\begin{array}{l} \boxed{(i)\ \sum_j n_j q_j \simeq 0\,;} \quad \boxed{(ii)\ m_i \gg m_e} \\ \boxed{(iii)\ |\omega| \ll |\Omega_c|, |ck|\,;} \end{array}\right\} \quad (9)$$

(B) Nonlinear GKE - $\mathbf{E}$ & $\mathbf{B}$ [PPCF (2019); PST (2020)]

◎ $\boxed{f(\mathbf{x}, \mu, v_\parallel, t) = F + f_{pol}}$ $\quad (10)$

$$\left.\begin{array}{l} \bullet\ \boxed{F = \exp(\boldsymbol{\rho}\cdot\nabla_\perp)F_g,} \quad \boldsymbol{\rho} = \mathbf{b}\times\mathbf{v}/\Omega_c \\ \mathbf{x} = \mathbf{X} + \boldsymbol{\rho}, \quad \mathbf{X} \Rightarrow \mathbf{x} \end{array}\right\} \quad (11)$$

(PST2020) unexpanded form

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$\circledcirc$ $\left[\partial_t + \dot{\underset{\sim}{X}} \cdot \underset{\sim}{\nabla} + \dot{V}_{\parallel} \partial_{V_\parallel}\right] F_g(\underset{\sim}{X}, \mu, V_\parallel, t) = 0$ (12)

$$\text{(13)}\left\{\begin{array}{l}
\circ\ \dot{\underset{\sim}{X}} = V_\parallel \underset{\sim}{B}_g^* / B_{g\parallel}^* + \underset{\sim}{V}_B + \underset{\sim}{V}_E \\
\circ\ \dot{V}_\parallel = (\underset{\sim}{B}_g^* / B_{g\parallel}^*) \cdot \left[\frac{q}{m} \langle \underset{\sim}{E} \rangle - \mu \underset{\sim}{\nabla} \langle B_g \rangle_* \right] \\
\circ\ \underset{\sim}{B}_g^* = \underset{\sim}{B}_g + (V_\parallel B / \Omega_c) \underset{\sim}{\nabla} \times \underset{\sim}{b}_g \\
\circ\ \underset{\sim}{B}_g = \langle \underset{\sim}{B} \rangle,\ \langle \cdots \rangle : \text{gyroaveraging};\ \underset{\sim}{b}_g = \underset{\sim}{B}_g / B_g \\
\circ\ B_{g\parallel}^* = \underset{\sim}{B}_g^* \cdot \underset{\sim}{b}_g \\
\circ\ \underset{\sim}{V}_B = (\mu B / \Omega_c B_{g\parallel}^*)\, \underset{\sim}{b}_g \times \underset{\sim}{\nabla} \langle B_g \rangle_* \\
\circ\ \underset{\sim}{V}_E = c \langle \underset{\sim}{E}_\perp \rangle \times \underset{\sim}{b}_g / B_{g\parallel}^* \\
\circ\ \langle \cdots \rangle_* : \text{gyroaveraging at } \rho/\sqrt{2}.
\end{array}\right.$$

$\circledcirc$ $f_{pol} = \frac{q}{m}\left[1 - \exp(\underset{\sim}{\rho} \cdot \underset{\sim}{\nabla}_\perp) J_0\right] \phi \frac{1}{B} \frac{\partial F}{\partial \mu}$ (14)

$\circ$ $J_0(k_\perp \rho)$, $k_\perp^2 = -\nabla_\perp^2$, $\underset{\sim}{\nabla}_\perp \phi = -\underset{\sim}{E}_\perp$

$\circledcirc$ $n_j, \underset{\sim}{P}_j, \partial_t J_{\parallel j}$: Assume $|k_\perp \rho|^2 < 1$ for now

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◎ $n = N + n_{pol}$ (15)

(16) $\begin{cases} \circ\ N = \langle F \rangle_v \\ \circ\ \boxed{n_{pol} = -\nabla \cdot [(Nq/m\Omega_c^2) E_\perp]} \end{cases}$

◎ $P = P_g + P_{pol}$ (17)

$\left.\begin{array}{l} \circ\ P_g = m \langle v v F \rangle_v \\ \circ\ \boxed{P_{pol} = -(3/4)\, I\, \nabla \cdot [(N q \rho_t^2/2) E_\perp]} \end{array}\right\}$ (18)

$\Rightarrow$ ◎ $\boxed{E_\perp = E_{\perp 0} + E_{\perp 1}}$

$\circ\ \boxed{E_{\perp 0} = -(u_{i\perp} \times B/c) + [\nabla \cdot P_{gi}]_\perp / (N_i q_i)}$

$\circ\ \boxed{E_{\perp 1} = -\frac{3}{4 N_i} \nabla_\perp \left[\nabla \cdot \frac{N_i}{2} \rho_{ti}^2 E_{\perp 0}\right]}$

◎ NLGKE $\Rightarrow$ parallel momentum conservation

o "electrons e.g."

(22) $4\pi \frac{\partial}{\partial t} J_{\parallel e} = \omega_{pe}^2 E_\parallel - \left(\frac{4\pi q}{m}\right)_e \{ (b \cdot \nabla) P_\parallel + (P_\parallel - P_\perp) \nabla \cdot b$

$+ m_e \nabla \cdot \langle v_\parallel (v_E + v_B + v_K) F_{ge} \rangle_v \}$

- $$\boxed{c^2\left[\nabla_\perp^2 E_\parallel - (\underset{\sim}{b}\cdot\underset{\sim}{\nabla})(\underset{\sim}{\nabla}\cdot \underset{\sim}{E}_\perp)\right] = 4\pi\,\partial J_\parallel/\partial t}$$

$\Rightarrow$ $\boxed{E_\parallel = E_{\parallel 0} + E_{\parallel 1}}$ ⟸ Only electrons for now

- $$\boxed{\begin{aligned} E_{\parallel 0} = \frac{1}{N q_e}\Big[&(\underset{\sim}{b}\cdot\underset{\sim}{\nabla})P_{g\parallel} + (P_{g\parallel} - P_{g\perp})\,\underset{\sim}{\nabla}\cdot\underset{\sim}{b} \\ &+ m\,\underset{\sim}{\nabla}\cdot\langle v_\parallel(\underset{\sim}{V}_E + \underset{\sim}{V}_B + \underset{\sim}{V}_\kappa)F_g\rangle_v\Big]_e \\ &- d_e^2(\underset{\sim}{b}\cdot\underset{\sim}{\nabla})(\underset{\sim}{\nabla}\cdot\underset{\sim}{E}_{\perp 0})\end{aligned}} \quad \boxed{d_e^2 = c^2/\omega_{pe}^2}$$

- $\underset{\sim}{V}_\kappa = (v_\parallel^2 B/\Omega B_{g\parallel}^*)\,\underset{\sim}{\nabla}\times\underset{\sim}{b}$

- $$\boxed{(d_e^2\nabla_\perp^2 - 1)E_{\parallel 1} = -d_e^2\nabla_\perp^2 E_{\parallel 0}}$$

$\Rightarrow$ For Alfvén waves : $|d_e^2\nabla_\perp^2| = |k_\perp^2\rho_i^2|\,|\frac{m_e}{m_i\beta_i}| \ll 1$

$\Rightarrow$ $E_{\parallel 1}$ calculated algebraically ($E_\parallel$)

$\Rightarrow$ For tearing modes : PDE near the singular surfaces ($|\underset{\sim}{b}\cdot\underset{\sim}{\nabla}| \simeq 0$)

- $\partial_t J_{\parallel i}$ can be included perturbatively ($|m_e| \ll m_i$)

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(III) Analytical Validation

$\Rightarrow$ Linear KAW dispersion relation

⊙ Single wave $(\omega, \underset{\sim}{k})$ in a uniform plasma

∘ $\underset{\sim}{B} = B_0 \hat{z}$

⊙ $\underset{\sim}{k} = (k_\perp, 0, k_\parallel)$, $\delta \underset{\sim}{E} = (\delta E_1, \delta E_2, \delta E_\parallel)$

$\delta \underset{\sim}{B} = (\delta B_1, \delta B_2, \delta B_\parallel)$, $F_g = F_{g0} + \delta F_g$; etc.

⊙ $|k_\perp| \gg |k_\parallel|$, $1 \gg \beta_i \sim \beta_e \gg m_e/m_i$

$\Rightarrow |\omega| \approx |k_\parallel V_A| \ll |k V_A| \Rightarrow$ suppress fast wave

$\Rightarrow \delta B_\parallel \propto \delta E_2 \approx 0$

(20) $\Rightarrow$ ⊙ $$k_\parallel \delta E_1 \left[1 - \frac{\omega_A^2}{\omega^2}\left(1 + \frac{3}{4} b_i\right)\right] = -\frac{\omega_A^2}{\omega^2}\left(1 + \frac{3}{4} b_i\right) k_\perp \delta E_\parallel \quad (22)$$

∘ $\omega_A^2 = k_\parallel^2 V_A^2$, $b_i = k_\perp^2 \rho_i^2 / 2$

(21), (22) $\Rightarrow$ ⊙ $$(1 + k_\perp^2 d_e^2) \delta E_\parallel = (i k_\parallel \delta P_{e\parallel} / n_0 q_e) + k_\parallel k_\perp d_e^2 \delta E_1 \quad (23)$$

(12) $\Rightarrow$ ∘ $$\delta P_{e\parallel} = -i\left[1 + \alpha_e^2 (1 - 2\alpha_e^2 + i\delta_e)\right](n_0 q_e / k_\parallel) \delta E_\parallel$$

∘ $\alpha_e = |\omega / k_\parallel v_{te}|$, $\delta_e = \sqrt{\pi}\, \alpha_e \exp(-\alpha_e^2)$

$$\Rightarrow \frac{\omega^2}{\omega_A^2} = 1 + \frac{3}{4} b_i + \frac{\tau b_i}{1 - 2\alpha_e^2 + i\delta_e}, \quad \tau = T_e / T_i$$

[Hasegawa + Chen, 1976]

(IV) Numerical Simulations

◎ Single wave : $\delta \underset{\sim}{E} = \delta \hat{E}(t) \exp[i \hat{k} \cdot \underset{\sim}{X}] + c.c;$ etc

(A) Linear $\underset{\sim}{B} = B_0 \hat{\underset{\sim}{z}}$, $\underset{\sim}{k} = (k_\perp, 0, k_\parallel)$

◎ $\rho_{m0} \frac{\partial}{\partial t} \delta \hat{u}_{i\perp} = \frac{1}{c} \delta \hat{\underset{\sim}{J}} \times \underset{\sim}{B}_0 - [\underset{\sim}{\nabla} \cdot \delta \hat{\underset{\approx}{P}}]_\perp$

$\underset{\sim}{\nabla} = i \underset{\sim}{k}$

◎ $\delta \hat{\underset{\sim}{J}} = \frac{c}{4\pi} i \underset{\sim}{k} \times \delta \hat{\underset{\sim}{B}}$

(20) ⇒ ◎ $\delta \hat{\underset{\sim}{E}}_\perp = (1 + \frac{3}{4} b_i) [- \delta \hat{\underset{\sim}{u}}_{i\perp} \times \frac{B_0}{c} + i \underset{\sim}{k}_\perp \cdot \delta \underset{\approx}{P}_{gi} / (n_0 q_i)]_\perp$

(21) ⇒ ◎ $\delta \hat{E}_\parallel (1 + k_\perp^2 d_e^2) = i k_\parallel \delta \hat{P}_{e\parallel} / (n_0 q_e) + k_\parallel k_\perp d_e^2 \delta E_\perp$

∘ $\delta \hat{P}_{e\parallel} = m_e \langle v_\parallel^2 \delta \hat{F}_e \rangle_v$

∘ $\partial_t \delta \hat{F}_e = - i k_\parallel v_\parallel \delta \hat{F}_e - \left(\frac{q}{m}\right)_e \delta \hat{E}_\parallel \frac{\partial F_{0e}}{\partial v_\parallel}$

◎ $\frac{\partial}{\partial t} \delta \underset{\sim}{B} = - c i \underset{\sim}{k} \times (\delta \hat{\underset{\sim}{E}}_\perp + \delta \hat{E}_\parallel \underset{\sim}{b})$

◎ Again, we let $\delta E_2 \propto \delta B_\parallel \simeq 0$ to suppress fast wave

◎ 2nd-order Runge-Kutta

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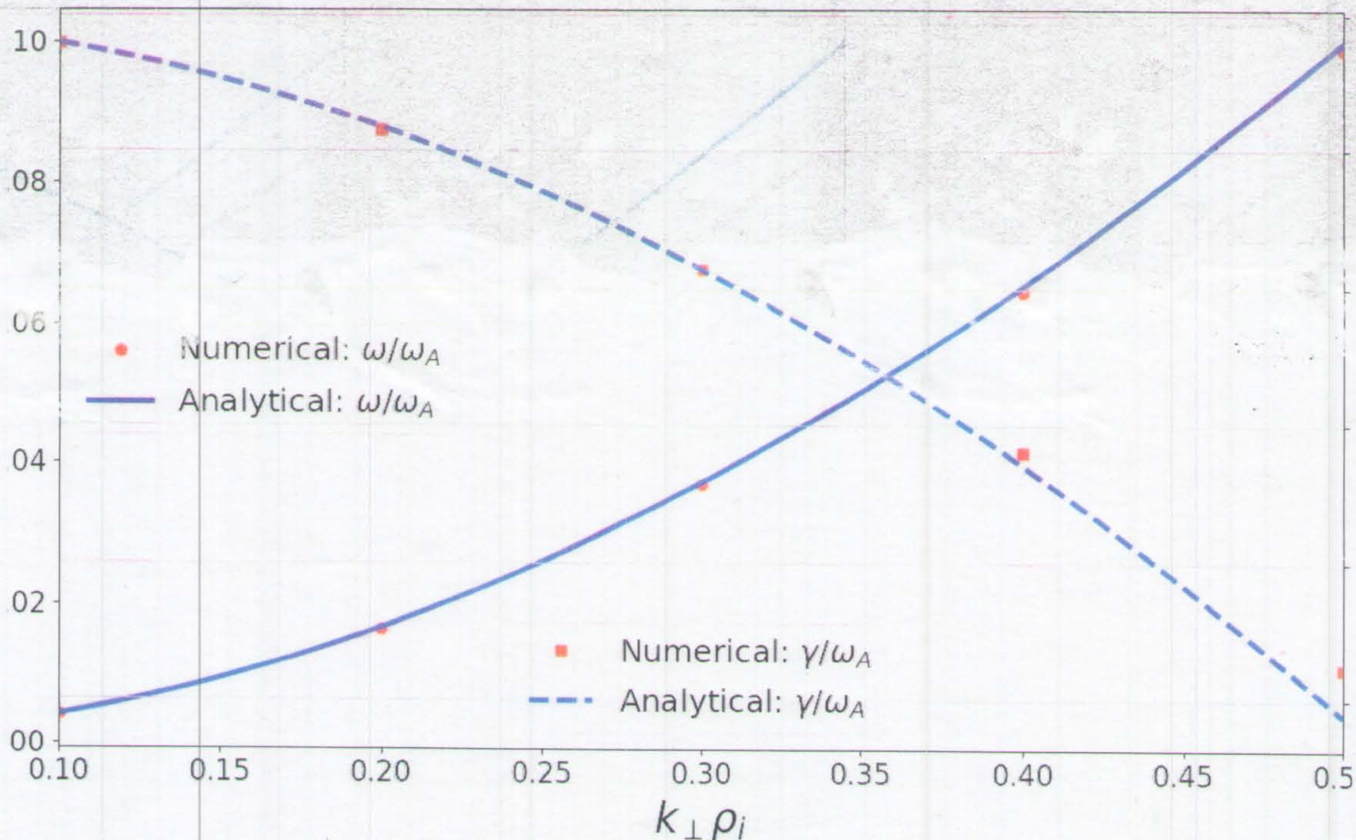

Numerical: ω/ω_A
Analytical: ω/ω_A
Numerical: γ/ω_A
Analytical: γ/ω_A
$k_\perp \rho_i$

(B) Nonlinear Simulations

$\Rightarrow$ Single wave model $\Rightarrow$ check wave trapping of resonant electrons

◎ $m_e \dot{v}_\parallel = q_e \delta \hat{E}_\parallel \cos(k_\parallel z - \omega t + k_\perp x_0)$

- $\dot{z} = v_\parallel$
- $k_\parallel z - \omega t + k_\perp x_0 \Rightarrow \Theta - \frac{\pi}{2}$

$$\Rightarrow \boxed{\ddot{\Theta} = \frac{q_e k_\parallel \delta \hat{E}_\parallel}{m_e} \sin \Theta}$$

$\Rightarrow$ ∘ trapping width : $|\Delta v_\parallel| = |2 q_e \delta \hat{E}_\parallel / m_e k_\parallel|^{1/2}$

∘ bounce frequency : $\omega_b = |q_e k_\parallel \delta \hat{E}_\parallel / m_e|^{1/2}$

◎ (i) $|\gamma_L|^2 \gg |\omega_b|^2 \Rightarrow$ linear damping

(ii) $|\omega_b|^2 \gg |\gamma_L|^2 \Rightarrow$ trapping dominates $\Rightarrow$ no Landau damping

◎ Simulation Parameters:

- $\delta \hat{e}_\parallel = \frac{e \delta \hat{E}_\parallel}{k_\parallel T_e} = 0.03$, $k_\perp \rho_i = 0.3$, $\beta = 10^{-2}$, $k_\parallel / k_\perp = 0.1$
- $\tau = 1 \Rightarrow |\omega_b| \simeq 0.74 \gg |\gamma_L| \sim O(10^{-2})$ (ii)

◎ δf scheme $\Rightarrow \delta F_{ge}$ nonlinear

◎ 32800 Marker particles • 64 grid points

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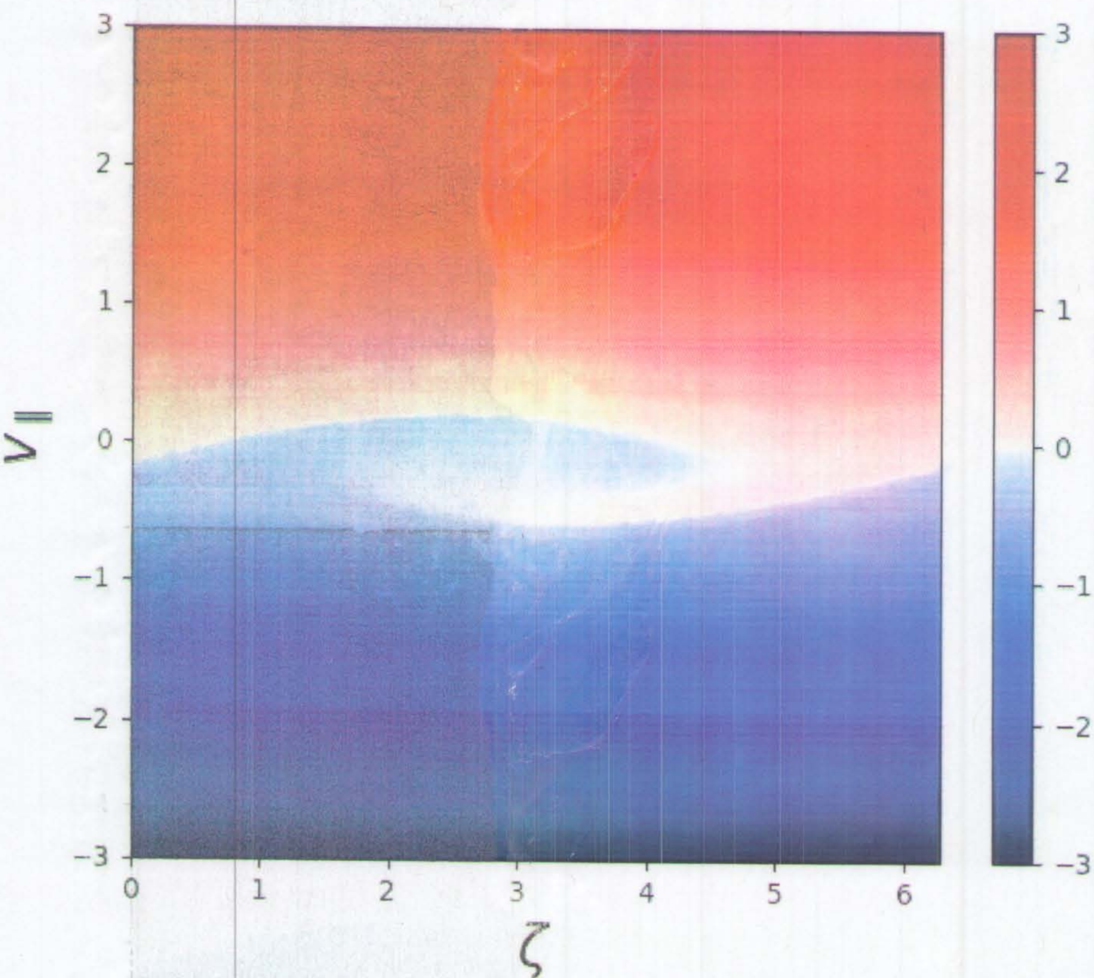
Simulation: $|\Delta V_{\parallel}| \simeq 0.8$ Theory: $|\Delta V_{\parallel}| \simeq 0.69$

$\omega_b \simeq 0.8$ $\omega_b \simeq 0.74$



[illegible]

(VI) Summary & Discussions

◎ GK-E&B: A new and promising model for simulating low-frequency e.m. fluctuations in general magnetically confined plasmas.

◎ It contains kinetic physics: microscopic finite-ρ_i effects and wave-particle interactions!

◎ Computationally easier than GK models using potentials (esp., for Alfvén waves)

◎ Analytical and numerical validations carried out in uniform plasmas

◎ More extensions and applications in laboratory and nature plasmas remain.

GK-E&B

RLS

$\frac{1}{2}$

P8 $\frac{3}{16}$

Remarks:

(i) electron force balance : Hall MHD

(o) valid in $\perp$ direction for $|\omega| \ll |\Omega_{ce}| \Rightarrow$ contains $\omega > \Omega_{ci}$ high-frequency (whistler waves): unnecessary for ($|\omega| < |\Omega_{ci}|$) e.m. waves: computationally and physics complexities!

(o) parallel (massless electron) force balance $\Rightarrow$ removes inertia $\boxed{\text{electron}}$ and, hence, wave-electron interaction $\Rightarrow$ incorrect $E_{\parallel}$.

(ii) In evaluating $E_{\parallel}$, the $(d_e^2 \nabla_\perp^2) E_{\parallel}$ term is negligible (or treated perturbatively) for Alfvén waves ($|\tau b_i| \ll 1$)

(tearing modes ??) (Yes, narrow electron current layer near the m.r.s.)

GK-E&B Simulation Model

Lu
2001
PPPL

Summary

(I) Assumptions

(i) low frequency $|\omega| \ll \Omega_i,\ |ck|$

(ii) λ (wavelength) $\gg \lambda_D$ (Debye length)

$\Rightarrow \sum_j n_j q_j \simeq 0$ quasi-neutrality

(iii) $m_i \gg m_e$

(II) Field equations

(i) Total perpendicular momentum conservation ($\perp \underset{\sim}{B}$)

$$\boxed{\frac{\partial}{\partial t}(\rho_{mi} \underset{\sim}{u}_{i\perp}) = -\left[\underset{\sim}{\nabla} \cdot \underset{\approx}{P}\right]_\perp + \underset{\sim}{J} \times \underset{\sim}{B}/c}$$

(ii) $\underset{\approx}{P} = \sum_j \underset{\approx}{P}_j$, $\underset{\approx}{P}_j = m_j \langle \underset{\sim}{v}\underset{\sim}{v} f_j \rangle_v$, $\rho_{mi} = m_i \langle f_i \rangle_v$
$= m_i n_i$

(iii) f_j given by NLGKE (later)

(iv) $\underset{\sim}{J} \simeq (c/4\pi)\, \underset{\sim}{\nabla} \times \underset{\sim}{B}$

(v) Ion perpendicular ($\perp \underset{\sim}{B}$) momentum balance

$+ |\omega| \ll |\Omega_i| \Rightarrow$

$$\boxed{\underset{\sim}{E}_\perp \simeq -\underset{\sim}{u}_i \times \underset{\sim}{B}/c + [\underset{\sim}{\nabla} \cdot \underset{\approx}{P}_i]_\perp / q_i n_i}$$

S-2 GK-E&B

(vi) Ampere's Law + Faraday's Law $\Rightarrow E_\parallel$

$$\Rightarrow \boxed{c^2\left[\nabla_\perp^2 E_\parallel - \underset{\sim}{b}\cdot\underset{\sim}{\nabla}\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{E}_\perp\right)\right] = 4\pi\frac{\partial J_\parallel}{\partial t}}$$

$\Rightarrow \partial J_\parallel/\partial t$ calculated from parallel momentum (along $\underset{\sim}{B}$) balance derived from NLGKE (later)

(vii) Faraday's Law

$$\boxed{\partial_t \underset{\sim}{B} = -c\,\underset{\sim}{\nabla}\times\left(\underset{\sim}{E}_\perp + \underset{\sim}{E}_\parallel\right)}$$

(IV) Nonlinear Gyrokinetic equations via $\underset{\sim}{E}$ and $\underset{\sim}{B}$.

(i) $\boxed{f = f_{pol} + T_g^{-1} F_g(\underset{\sim}{X}, \mu, v_\parallel)}$

(ii) $\dot{\underset{\sim}{X}} = \dot{\underset{\sim}{X}}[\underset{\sim}{E}, \underset{\sim}{B}]$, $\dot{v}_\parallel = \dot{v}_\parallel[\underset{\sim}{E}, \underset{\sim}{B}]$

[Chen et al., PPCF 2019]

[Chen et al., PS&T, 2020]

(iii) $\boxed{f_{pol} = f_{pol}[\underset{\sim}{E}_\perp]}$

Cancellation Problem

Summary

T.S. Hahm
Dec. 2020

(I) ($A_{\parallel}$–$P_{\parallel}$) formulation

$$\nabla^2 \delta A_{\parallel} = \frac{\omega_{pe}^2}{c^2}\delta A_{\parallel} - \frac{4\pi q}{mc}\langle f P_{\parallel}\rangle_v$$

$\langle \cdots \rangle_v$ contains statistical error due to velocity-space integration : error Δ_s

$$\Rightarrow \left[\nabla^2 + \frac{\omega_{pe}^2}{c^2}\Delta_s\right]\delta A_{\parallel} = -\frac{4\pi}{c}\delta J_{\parallel}$$

$$\Rightarrow |\Delta_s| \ll \left|\frac{c^2\nabla_\perp^2}{\omega_{pe}^2}\right| = d_e^2\nabla_\perp^2$$

(II) GK-E+B formulation

o $|\alpha_e|^2 = |\omega/k_{\parallel}v_{te}|^2 \ll 1$

$$\Rightarrow \omega_{pe}^2 E_{\parallel}[O(\alpha_e^2) + d_e^2\nabla_\perp^2 + \Delta_s'] = c^2(\underline{b}\cdot\underline{\nabla})(\underline{\nabla}\cdot\underline{E_{1o}}).$$

o $|\Delta_s'|$ satistical error in $P_{e\parallel}$ velocity integral

$$\Rightarrow |\Delta_s'| \ll |\alpha_e|^2 \sim |m_e/m_i\beta_e| \text{ for Alfvén waves}$$

$(10^{-3} - 10^{-1})$

$$\Rightarrow |d_e^2\nabla_\perp^2| \simeq |k_\perp^2\rho_i^2||d_e^2/\rho_i^2| \sim |k_\perp\rho_i|^2|m_e/m_i\beta_i|$$

$$\simeq |k_\perp\rho_i|^2|\alpha_e^2|.$$

$\Rightarrow$ much relaxed numerical accuracy requirement!

On the so-called "cancellation" problem

陈骝 Dec 2020

(I) The original problem

It arises from the solving $\delta A_{\parallel}$ in the parallel Ampere's Law via the $P_{\parallel}$ formulation.

$$\nabla^2 \delta A_{\parallel} = -4\pi \delta J_{\parallel}/c \quad (1)$$

$$p_{\parallel} = m v_{\parallel} + q\,\delta A_{\parallel}/c \quad \cdots \quad (1)'$$

$$\Rightarrow \nabla^2 \delta A_{\parallel} = \frac{\omega_{pe}^2}{c^2}\delta A_{\parallel} - \frac{4\pi}{m}\frac{q}{c}\,\delta J_p \quad (2)$$

$$\delta J_p = \langle q f P_{\parallel} \rangle_v \quad (3)$$

δJ_p is computed numerically, and $\delta A_{\parallel}$ is then solved via Eq. (2).

When computed numerically

$$\delta J_{p,cmp} = m\,\delta u_{\parallel} + \frac{n q}{c}\,\delta A_{\parallel}[1+\Delta_1] \quad (4)$$

Here, Δ_1 denotes numerical inaccuracy. That is, with infinite accuracy, we have $\Delta = 0$ and one recovers analytical result.

(4) into (2)

$$\Rightarrow \nabla^2 \delta A_{\parallel} = -\frac{4\pi \delta J_{\parallel}}{c} + \frac{\omega_{pe}^2}{c^2}\delta A_{\parallel}\cdot\Delta_1 \quad (5)$$

Thus, to obtain accurate numerical solution, we need to have $|\Delta_1| \ll |d_e^2 \nabla_\perp^2| \lesssim 10^{-7}$ (6)

For $|\nabla_\perp| \sim 1/a$,

$$|\Delta| \ll |d_e^2 / a^2|$$ very stringent!

$\rightarrow G_e/L_0$

(II) In Gk.-E+B

- For Alfvén waves $|d_e^2 \nabla_\perp^2| \ll 1$.

$$\Rightarrow E_\parallel = E_{\parallel 0} + E_{\parallel 1} \qquad (7)$$

$$-\omega_{pe}^2 E_{\parallel 0} = -\frac{4\pi q_e}{m_e} \underset{\sim}{b} \cdot \underset{\sim}{\nabla} P_{e\parallel} + c^2 (\underset{\sim}{b} \cdot \underset{\sim}{\nabla})(\underset{\sim}{\nabla} \cdot \underset{\sim}{E}_{\perp 0}). \qquad (8)$$

$$E_{\parallel 1} = d_e^2 \nabla_\perp^2 E_{\parallel 0}. \qquad (9)$$

- Now $P_{e\parallel}$ is computed numerically via particles (statistical noise)

$$\underset{\sim}{b} \cdot \underset{\sim}{\nabla} P_{e\parallel} = n_0 q_e E_\parallel [1 + O(\alpha_e^2) + \Delta_2]. \qquad (10)$$

- $|\alpha_e| = |\omega / k_\parallel v_{te}| < 1$ ◦ Δ_2: numerical inaccuracy.

◦ examine the $|v_A| / |v_{te}| < 1$ limit

$E_\parallel$ eqn. $\Rightarrow \omega_{pe}^2 E_\parallel [O(\alpha_e^2) + \Delta_2] = c^2 (\underset{\sim}{b} \cdot \underset{\sim}{\nabla})(\underset{\sim}{\nabla} \cdot \underset{\sim}{E}_{\perp 0})$. (11)

◦ thus, accuracy requires

$$|\Delta_2| \ll |\alpha_e|^2 \sim |m_e / m_i \beta_e| \simeq (10^{-2} - 10^{-1}).$$

- Note that also that for $|k_\perp \rho_i|^2 < 1$

$\Rightarrow |\alpha_e|^2 \gg |k_\perp^2 d_e^2|$. i.e., $E_{\parallel 1}$ can be neglected.

ⓒ In fact, for typical Alfvén waves, we can take

$$|k_\perp^2 d_e^2| \ll |\alpha_e|^2 \ll 1$$

and ignore $\underline{\underline{\delta E_\parallel 1}}$. (7)

In this case, we can readily show that the dispersion relation of KAW becomes

$$\frac{\omega^2}{\omega_A^2} = \left(1+\frac{3}{4}b_i\right)\left[1 + \frac{\tau b_i}{1-2\alpha_e^2+i d_e}\right]. \quad (8)$$

this is to be contrasted with Eq.(24) in Sc-PRA

$$\frac{\omega^2}{\omega_A^2} = \left(1+\frac{3}{4}b_i\right) + \frac{\tau b_i}{1-2\alpha_e^2+i d_e}. \quad (28)'$$

ⓓ Thus, in GK-E+B numerical accuracy requirement is

$$|\Delta x| \ll |\alpha_e|^2 = \left|\frac{\omega^2}{k_\parallel^2 v_A^2}\right| \approx \left|\frac{m_e/m_i}{\beta}\right|.$$